

# The diagonal lemma as the formalized Grelling paradox<sup>1</sup>

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Gödel's diagonal lemma (which is often referred to as fix-point or self-referential lemma) summarizes very succinctly the ability of first-order arithmetic to 'talk about itself', a crucial property of this system that plays a key role in the proof of Gödel's incompleteness theorem and in those of the theorems of Tarski and Church on the undefinability of truth and undecidability of provability respectively. In fact, with the representability of provability at hand, these three main limitative theorems of logic can be considered to be simple applications of the lemma (cf. e.g. [2] pp. 227–231). Due to this central role, the proof of the lemma could shed light on the very essence of these fundamental theorems. In spite of the fact that it is common knowledge that Gödel's proof of the incompleteness theorem is closely related to the Liar paradox, the proof of the lemma as it is presented in textbooks on logic is not self-evident to say the least. Indeed, in the *Handbook of Proof Theory*, the proof of the lemma is introduced by the following remark (see [1], p.119): 'This proof [is] quite simple but rather tricky and difficult to conceptualize.' Or to quote another opinion, 'The brevity of the proof does not make for transparency; it has the aura of a magician's trick' (cf. [4], p. 1). It seems, therefore, that the words of a respected logician reflect a widespread attitude to the proof of the lemma (see [5]): '[This] result is a cornerstone of modern logic. [...] You would hope that such a deep theorem would have an insightful proof. No such luck. [...] I don't know anyone who thinks he has a fully satisfying understanding of why the Self-referential Lemma works. It has a rabbit-out-of-a-hat quality for everyone.'

In view of these remarks, we think that it is worth drawing attention to a possibility of making the proof of the lemma completely transparent by showing that it is simply a straightforward translation of the Grelling paradox into first-order arithmetic.<sup>2</sup>

## Notation

Our formal language is that of first-order arithmetic.  $Q$  stands for Robinson arithmetic while  $\omega$  is the set of natural numbers.  $g$  is any one of the standard Gödel numberings and  $Fm_n$  is the set of formulas with all free variables among the first  $n$  ones. For the sake of simplicity, we shall denote the closed terms corresponding to natural numbers by the numbers themselves. Further,  $N$  denotes the set of Gödel numbers of formulas in  $Fm_1$ . Finally, the result of substituting a term  $t$  for the only free variable of a formula  $\varphi \in Fm_1$  is denoted by  $\varphi(t)$ .

## Diagonal lemma

For any formula  $\varphi \in Fm_1$ , there is a sentence  $\lambda$  such that

$$Q \vdash \lambda \longleftrightarrow \varphi(g(\lambda)).$$

## Proof idea

First we show how to construct, out of Grelling's paradox, an ordinary language sentence that, on the one hand, says of itself that it has a given property, on the other hand, consists of components with easily identifiable formal first-order counterparts. The straightforward formalization of this ordinary language sentence leads to the desired formal sentence (as can be expected since the lemma is just about the existence of a first-order sentence that, informally speaking, says of itself that it has a given property).

As is well known, the Grelling paradox consists in the fact that the sentence

- (1) 'heterological' is heterological<sup>3</sup>

shares with the Liar sentence the remarkable property that its truth implies its own falsity and *vice versa*, i.e., in effect, says of itself that it is false. What is truly important is that, contrary to the Liar, this paradoxical sentence achieves self-reference without using an indexical.<sup>4</sup>

<sup>1</sup>To appear in the series *Collegium Logicum* of the Kurt Gödel Society in 2006.

<sup>2</sup>The train of thought below is an application of the ideas given in [6] to first-order arithmetic.

<sup>3</sup>An adjective is called *heterological* if the property denoted by the adjective does not hold for the adjective itself; e.g. 'long', 'German', 'monosyllabic' are heterological.

<sup>4</sup>As to the notion of heterologicality itself (which actually involves self-reference), its slightly modified version can also be expressed without using an indexical: replace 'it' by ' $x$ ' in (3) below.

Since our aim is to construct a sentence that (a) is not about an adjective but about a sentence and (b) instead of asserting its own falsehood, says of itself that it has an arbitrary (but fixed) property, we have to slightly modify (1) accordingly. In order to satisfy the first requirement, in place of an adjective  $A$ , we consider the open sentence ' $x$  is  $A$ '. Obviously, in this case, the transformation corresponding to the application of an adjective  $A$  to a linguistic object  $O$  will be the substitution of the name of  $O$ <sup>5</sup> for the variable  $x$  in the open sentence corresponding to  $A$ . Consequently, the sentence associated with the self-application of any adjective  $A$  in this way is " $x$  is  $A$ ' is  $A$ ". In particular, the counterpart of (1) is:

(2) ' $x$  is heterological' is heterological.

Note that the notion of heterologicality occurring here is already a property of open sentences with single variables. Since, on the one hand, to be heterological is to have the property that its application to itself yields a false sentence, on the other, as we noted above, in the case of sentences, 'applied to itself' means 'its name is substituted for the variable in it', for any open sentence  $x$  with a single variable, we have

(3)  $x$  is heterological just in case the sentence obtained by substituting the name of  $x$  for the variable in it is false.

Finally, if we replace 'being false' by 'having property  $p$ ', (2) and (3) together yield:

(4) the sentence obtained by substituting the name of ' $x$  is heterological' for the variable in it has property  $p$ .

It can directly be checked that this sentence indeed says of itself that it has property  $p$  (and says nothing else) since it is built up in such a way that if we perform the substitution described in it, then we obtain the sentence itself, which is stated to have property  $p$ .<sup>6</sup> Now, let  $s$  denote the open sentence between the quotation marks in (4):

the sentence obtained by substituting the name of  $x$  for the variable in it has property  $p$ .

Then, clearly, the whole sentence (4) is  $s('s')$ .<sup>7</sup> That is, the formalization process should consist of two steps. In the first step we have to find the formal version  $\eta \in Fm_1$  of  $s$ , and then the second step is obvious: the desired sentence  $\lambda$  will simply be  $\eta(g(\eta))$ .<sup>8</sup>

## Proof

Let  $\varphi \in Fm_1$  be arbitrary and let its informal counterpart be the open sentence ' $x$  has property  $p$ '.<sup>9</sup> Certainly,  $x(g(x))$  is the formal version of the phrase

the sentence obtained by substituting the name of  $x$  for the variable in it,

and hence  $\varphi(g[x(g(x))])$  is the formal version of  $s$ . Clearly,  $\varphi(g[x(g(x))])$  with a variable  $x$  running over formulas in  $Fm_1$ <sup>10</sup> is not a formula itself, it becomes a formula only if we replace the variable  $x$  by a formula. Therefore, we cannot continue the formalization process unless we find a formula that can play the role of  $\varphi(g[x(g(x))])$ , that is, a formula  $\eta \in Fm_1$  such that  $\eta(g(\psi))$  is provably equivalent in  $Q$  to  $\varphi(g[\psi(g(\psi))])$  for every  $\psi \in Fm_1$ , or equivalently (denoting the inverse of  $g$  by  $g^{-1}$ ), for any  $n \in N$ ,

$$Q \vdash \eta(n) \longleftrightarrow \varphi(g[g^{-1}(n)(n)]).$$

In order to find the appropriate formula  $\eta$ , let us consider the expression substituted into the formula  $\varphi$ , and define the function  $f : \omega \rightarrow \omega$  accordingly:

$$f(n) = g[g^{-1}(n)(n)] \text{ if } n \in N \text{ and } f(n) = 0 \text{ otherwise.}$$

<sup>5</sup>Following the common practice, we define the name of a linguistic object to be the object itself between quotation marks.

<sup>6</sup>J.N. Findlay used sentences of the same structure to examine informally the incompleteness theorem (cf. [3]).

<sup>7</sup>Mimicking the formal notation, in the case of any common language open sentence  $o$  having a single variable, we abbreviate the result of substituting a linguistic phrase  $q$  for the variable in  $o$  by  $o(q)$ .

<sup>8</sup>Gödel numbering is, of course, the formal counterpart of naming.

<sup>9</sup>It is obvious that the formulas in  $Fm_1$  are formal versions of open sentences with single variables asserting the possession of a property, and, taking into consideration only those informal concepts that have formal counterparts, the formalization of attributing a property to an object is the substitution of the formal name (i.e. the Gödel number) of the corresponding formal object for the only free variable of the formula that formalizes the open sentence asserting the possession of the property concerned.

<sup>10</sup>Recall that, in  $s$ , the variable  $x$  runs over the set of open sentences with single variables.

Since this function is obviously recursive and hence representable in  $Q$ , and, up to provable equivalence in  $Q$ , the result of substituting a representable function into a formula can also be expressed by a formula,<sup>11</sup> there is a formula  $\eta \in Fm_1$  such that, for any  $n \in N$ ,

$$(5) \quad Q \vdash \eta(n) \longleftrightarrow \varphi(f(n)).$$

Thus we have obtained what we need, we have shown that there exists an  $\eta \in Fm_1$  that can be considered to be the formal version of  $s$ . Now, all that remains to do is straightforward: it follows from (5) that, for every  $\psi \in Fm_1$ ,

$$Q \vdash \eta(g(\psi)) \longleftrightarrow \varphi(g[\psi(g(\psi))]),$$

which, in turn, choosing  $\psi$  to be  $\eta$ , yields

$$Q \vdash \eta(g(\eta)) \longleftrightarrow \varphi(g[\eta(g(\eta))]),$$

showing that the sentence  $\lambda = \eta(g(\eta))$  indeed has the desired property.<sup>12</sup>

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## References

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<sup>11</sup>By elementary first-order logic, it follows from the definition of representability that if a function  $f : \omega \longrightarrow \omega$  is represented in  $Q$  by a formula  $\mu \in Fm_2$ , then, for any  $\varphi \in Fm_1$  and  $n \in \omega$ ,

$$Q \vdash (\exists y)(\mu(n, y) \wedge \varphi(y)) \longleftrightarrow \varphi(f(n)).$$

<sup>12</sup>Perhaps it is worth noting that the informal version of the last step in the formal proof explains the reasons why  $s$  (the informal counterpart of  $\eta$ ) is suitable for constructing the appropriate self-referring sentence (4). Actually, by definition, for *any* open sentence  $o$  with a single variable,  $s(o')$  says that  $o(o')$  has property  $p$ . In the particular case when  $o$  is just  $s$ , we obtain:  $s(s')$  says that  $s(s')$  has property  $p$ .